The 7th Balkan Conference on Operational Research
"BACOR 05"
Constanta, May 2005, Romania

THE INTERIOR – POINT METHOD TO DETERMINE THE SYMMETRY OF THE POLYHEDRAL SETS

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Abstract
There is a variety of measures of symmetry (or asymmetry) for convex sets that have been studied over the years, see Grűnbaum [1] and Minkowski [3]. These measures arise naturally in the complexity theory of interior - point methods. In this work we present a method for computing an approximation of the symmetry and a symmetry point of the polyhedral sets.

Keywords: *interior – point method, approximation, convex set, symmetry*

1. INTRODUCTION

Given a closed convex set S and a point x, define the symmetry of S about x as follows:

$$sym(x,S) = \max\{\alpha \geq 0 \mid x + \alpha(x-y) \in S, (\forall) y \in S\} \tag{1.1}$$

which intuitively states that $sym(x,S)$ is the largest scalar α such that every point $y \in S$ can be reflected through x by the factor α and still lie in S. The symmetry value of S then is:

$$sym(S) = \max_{x \in S} sym(x,S) \tag{1.2}$$

and x^* is a symmetry point of S if x^* achieves the above supremum (also called a "critical point" in [1],[2] and [3]). S is symmetric if $sym(S) = 1$.

We present a simple example. Let be $a,b \in R, a < b$ and $S = [a,b]$. We show how we can determinate $sym(x,S), x \in S$ and $sym(S)$. Let be $\alpha \geq 0$ so that $x + \alpha(x-y) \in S$,

$(\forall)y \in S$. Results that $a \le x + \alpha(x-b) \le x + \alpha(x-y) \le x + \alpha(x-a) \le b$ so $\alpha \le \dfrac{a-x}{x-b}$ and $\alpha \le \dfrac{b-x}{x-a}$. Then $\alpha \in \left[0, \min\left\{\dfrac{a-x}{x-b}, \dfrac{b-x}{x-a}\right\}\right]$. Let be

$$f(x) = \frac{a-x}{x-b} - \frac{b-x}{x-a} = \frac{(b-a)(a+b-2x))}{(x-b)(x-a)}.$$

Results that

$$sym(x,S) = \min\left\{\frac{a-x}{x-b}, \frac{b-x}{x-a}\right\} = \begin{cases} \dfrac{a-x}{x-b} & if \quad x \in \left[a, \dfrac{a+b}{2}\right] \\ \dfrac{b-x}{x-a} & if \quad x \in \left(\dfrac{a+b}{2}, b\right] \end{cases} \tag{1.3}$$

We will compute $sym(S) = \max\limits_{x \in S} sym(x,S)$. Using (1.3), let be $f_1 : \left[a, \dfrac{a+b}{2}\right] \to R$, $f_1(x) = \dfrac{a-x}{x-b}$ and $f_2 : \left(\dfrac{a+b}{2}, b\right] \to R$, $f_2(x) = \dfrac{b-x}{x-a}$. It is obvious that $0 \le f_1(x) \le 1 = f_1\left(\dfrac{a+b}{2}\right)$ and $0 \le f_2(x) < 1$. So $sym(S) = \max\limits_{x \in S} sym(x,S) = 1$ (S is symmetric) and $x^* = \dfrac{a+b}{2}$ is a symmetry point of S.

Symmetric convex sets play an important role in convexity theory. There are many other geometric properties of convex bodies S that are also connected to $sym(S)$. Notice that $sym(x,S)$ and $sym(S)$ are invariant under invertible affine transformation and change in norm. The relevance of $sym(x,S)$ has been revived in the complexity theory of interior - point methods for convex optimization, see Nesterov and Nemirovskii [4] and Renegar [5].

2. GENERAL PROPERTIES OF $sym(x,S)$ AND $sym(S)$

We make the following assumption: S is a convex body, i.e., S is a nonempty closed bounded convex set with a nonempty interior. We assume that S has an interior. Notice that the definition of $sym(x,S)$ given in (1.1) is equivalent to the following “set-containment” definition:

$$sym(x,S) = \sup\{\alpha \ge 0 | \alpha(x-S) \subseteq (S-x)\} \tag{2.1}$$

The general properties from the $sym(x,S)$ will be shown without demonstration in the following propositions.

Proposition 2.1 If S is a nonempty closed bounded convex set with a nonempty interior, $sym(\cdot,S) : S \to [0,1]$ is a continuous quasiconcave function.

Proof. See [1]

Proposition 2.2 If S is a nonempty closed bounded convex set with a nonempty interior, a symmetric set centered at the origin, and $\|\cdot\|_S$ denote the norm induced by S. Then, for every $x \in S$, $sym(x,S) = \dfrac{1-\|x\|_S}{1+\|x\|_S}$.

Proof. See [1]

Proposition 2.3 If S is a nonempty closed bounded convex set with a nonempty interior, a symmetric set centered at the origin. Then $sym(\cdot,S)$ is a logconcave function in S.

Proof. See [1]

Proposition 2.4 Let $S,T \subset R^n$ be convex bodies, and let be $x \in S$ and $y \in T$. Then:

If $x \in S \cap T$,

$$sym(x, S \cap T) \geq \min\{sym(x,S), sym(x,T)\} \tag{2.2}$$

$$sym(x+y, S+T) \geq \min(sym(x,S), sym(y,T)\} \tag{2.3}$$

$$sym((x,y), S \times T) = \min\{sym(x,S), sym(y,t)\} \tag{2.4}$$

Let be $A(\cdot)$ be an affine transformation. Then:

$$sym(A(x), A(S)) \geq sym(x,S) \tag{2.5}$$

with equality if $A(\cdot)$ is invertible.

Proof. See [1]

3. COMPUTING A SYMMETRY POINT OF S WHEN S IS POLYHEDRAL

Our interest lies in computing an ε-approximate symmetry point of S, which is a point $x \in S$, that satisfies: $sym(x,S) \geq (1-\varepsilon) sym(S)$. We assume that $S = \{x \in R^n \mid Ax \leq b, A \in R^{m \times n}, b \in R^m\}$ i.e., S is polyhedral. We present a method for computing an ε-approximate symmetry point of S. The method involves solving $m+1$ linear programs each of which involves m linear inequalities in n unrestricted variables.

Define the following scalar quantities δ_i^*, $i = 1, \ldots, m$:

$$\delta_i^* = \max_x (-A_i x) \tag{3.1}$$
$$Ax \leq b$$

and notice that $b_i + \delta_i^*$ is the range of $A_i x$ over $x \in S$ if the ith constraint is not strictly redundant on S. We compute δ_i^*, $i = 1, \ldots, m$ by solving m linear programs whose feasible region is exactly S. The following proposition shows that if we know δ_i^*, $i = 1, \ldots, m$, then we are able to compute $sym(x,S)$ for any $x \in S$.

Proposition 3.1 Let $S = \{x \in R^n \mid Ax \leq b, A \in R^{m \times n}, b \in R^m\}$ be given. For each $x \in S$,

$$sym(x,S) = \min_{i=1,\ldots,m} \left\{ \frac{b_i - A_i x}{\delta_i^* + A_i x} \right\} \tag{3.2}$$

Proof. Let be $\alpha = sym(x,S)$ and $\beta = \min_{i=1,\dots,m}\left\{\frac{b_i - A_i x}{\delta_i^* + A_i x}\right\}$. We will proof that $\alpha = \beta$.

For all $y \in S$, $x + \alpha(x-y) \in S$, so $A(x+\alpha(x-y)) \le b$ and so $A_i x + \alpha A_i x + \alpha(-A_i y) \le b_i$, $i = 1,\dots,m$. This implies that $A_i x + \alpha A_i x + \alpha\delta_i^* \le b_i$ and $\alpha \le \min_{i=1,\dots,m}\left\{\frac{b_i - A_i x}{\delta_i^* + A_i x}\right\}$ and so $\alpha \le \beta$

On the other hand for all $y \in S$ and $i = 1,\dots,m$ we have: $\beta \le \frac{b_i - A_i x}{\delta_i^* + A_i x}$, for all $i = 1,\dots,m$, so $b_i - A_i x \ge \beta(A_i x + \delta_i^*) \ge \beta(A_i x - A_i y)$, $i = 1,\dots,m$. Thus $A_i x + \beta A_i x + \beta(-A_i y) \le b_i$ and so $A_i(x + \beta(x-y)) \le b$, $i = 1,\dots,m$ which implies that $\alpha \ge \beta$. Thus $\alpha = \beta$ that means $sym(x,S) = \min_{i=1,\dots,m}\left\{\frac{b_i - A_i x}{\delta_i^* + A_i x}\right\}$.

Proposition 3.1 can be used to create another single linear program to compute $sym(S)$ as follows. Let be $\delta^* = (\delta_1^*,\dots,\delta_m^*)$ and consider the following linear program that uses δ^* in the data:

$$\max_\theta(\theta) \tag{3.3}$$

$$Ax + \theta(\delta^* + b) \le b .$$

Proposition 3.2 Let be (x^*,θ^*) be an optimal solution of the linear program (3.3). Then x^* is a symmetry point of S and $sym(S) = \frac{\theta^*}{1-\theta^*}$.

Proof. Suppose that $(\bar{x},\bar{\theta})$ is a feasible solution of (3.3). Then $A_i\bar{x} + \bar{\theta}(\delta_i^* + b_i) \le b_i$, $i = 1,\dots,m$.

Because $\delta_i^* = \max_x(-A_i x)$ and $A_i\bar{x} \le b_i$ then $\delta_i^* \ge -A_i\bar{x} \ge -b_i$, $i = 1,\dots,m$, so $\delta_i^* + b_i > 0$ and that implies $\bar{\theta} \le \frac{b_i - A_i\bar{x}}{\delta_i^* + b_i}$, $\frac{1}{\bar{\theta}} \ge \frac{\delta_i^* + b_i}{b_i - A_i\bar{x}}$, $\frac{1}{\bar{\theta}} - 1 \ge \frac{\delta_i^* + b_i}{b_i - A_i\bar{x}} - 1 = \frac{\delta_i^* + A_i\bar{x}}{b_i - A_i\bar{x}}$. We have $\frac{1-\bar{\theta}}{\bar{\theta}} \ge \frac{\delta_i^* + A_i\bar{x}}{b_i - A_i\bar{x}}$ and so $\frac{\bar{\theta}}{1-\bar{\theta}} \le \frac{b_i - A_i\bar{x}}{\delta_i^* + A_i\bar{x}}$. Then from Proposition 3.1 follows that $sym(\bar{x},S) \ge \frac{\bar{\theta}}{1-\bar{\theta}}$, which implies that $sym(S) \ge \frac{\theta^*}{1-\theta^*}$. The reverse inequality follows analogous.

From the propozitions 3.1 and 3.2 results the following method for computing $sym(S)$ and a symmetry point x^*:

Step 1 For $i = 1, \ldots, m$, approximately solve the linear program (3.1), stopping each linear program when a feasible solution $\bar{x}$ is computed for which the duality gap $\bar{g}$ satisfies $\bar{g} \le \frac{\varepsilon(b_i - A_i\bar{x})}{4.1}$. Set $\bar{\delta}_i = -A_i\bar{x}$.

Step 2 Let be $\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_m)$. Approximately solve the linear program

$$\max_{\theta}(\theta) \tag{3.4}$$

$$Ax + \theta(\bar{\delta} + b) \le b$$

stopping when a feasible solution $(\bar{x}, \bar{\theta})$ is computed for which the duality gap $\bar{g}$ satisfies $\bar{\theta} \ge (\bar{\theta} + \bar{g})\left(1 - \frac{\varepsilon}{4.1}\right)$. Then $\bar{x}$ will be an ε-approximate symmetry point of S and $\frac{\bar{\theta}(1-\varepsilon)}{1-\bar{\theta}} \le sym(S) \le \frac{\bar{\theta}}{1-\bar{\theta}}$.

Notice that this method requires that the LP solver computes primal and dual feasible points (or simply primal feasible points and the duality gap) at each of its iterations; such a requirement is satisfied, for example, by a standard feasible interior - point method.

In order to prove a complexity bound for the method, we will assume that S is bounded and has an interior.

A standard interior - point method for solving (3.1) uses Newton's method to compute successive β-approximate solutions for a decreasing sequence of values of barrier parameter $\mu > 0$. Is well know [6], [7] the following result:

Theorem 1 Suppose that $\beta = \frac{1}{4}$ and that (x^0, s^0, z^0) is a given β - approximate solution for the barrier parameter $\mu^0 > 0$. If $(\bar{x}, \bar{s}, \bar{z})$ is a β - approximate solution for the barrier parameter $\bar{\mu} \in (0, \mu^0)$ then such a solution can be computed in at most $\left[\left(2 + 4\sqrt{m}\right)\ln\frac{\mu^0}{\mu}\right]$ iterations of Newton method, and the duality gap associated with variables $(\bar{x}, \bar{s}, \bar{z})$ satisfy $\bar{g} \le \frac{5}{4}m\bar{\mu}$.

Proof. See [6].

Proposition 3.3 Let $\varepsilon \in (0, 0.1)$ be given, set $\bar{\varepsilon} = \frac{\varepsilon}{4.1}$, and suppose that Step 1 of the method is executed. Then $\bar{\delta} = (\bar{\delta}_1, \ldots, \bar{\delta}_m)$ satisfies $\delta_i^* - \bar{\varepsilon}(\delta_i^* + b_i) \le \bar{\delta}_i \le \delta_i^*$, $i = 1, \ldots, m$. Furthermore, for any given $x \in S$, let be $\theta = \min_{i=1,\ldots,m}\left\{\frac{b_i - A_i x}{\delta_i^* + A_i x}\right\}$. Then

$$\frac{\theta}{1-\theta}\left(\frac{1-3\bar{\varepsilon}}{1-\bar{\varepsilon}}\right) \le sym(x, S) \le \frac{\theta}{1-\theta} \tag{3.5}$$

Proof. For a given $i=1,\ldots,m$ let $\bar{g}$ denote the duality gap computed in the stopping criterion of Step 1 of the method. Then

$$\delta_i^* \ge \bar{\delta}_i \ge \delta_i^* - \bar{g} \ge \delta_i^* - \bar{\varepsilon}(\delta_i^* + b_i) \tag{3.6}$$

Adding b_i in all members of (3.6) implies that

$$(1-\bar{\varepsilon})(\delta_i^* + b_i) \le (\bar{\delta}_i + b_i) \le (\delta_i^* + b_i) \tag{3.7}$$

For a given $x \in S$ let be $\alpha = sym(x,S)$ and $\bar{\theta} = \min_{i=1,\ldots,m}\left\{\frac{b_i - A_i x}{\delta_i^* + A_i x}\right\}$. Then from Proposition 3.1 we have

$$\alpha = \min_{i=1,\ldots,m}\left\{\frac{b_i - A_i x}{\delta_i^* + A_i x}\right\} = \frac{\bar{\theta}}{1-\bar{\theta}} \tag{3.8}$$

Notice that $\bar{\delta}_i \le \delta_i^*$ for all i, whereby $\theta \ge \bar{\theta}$, which implies that $\alpha = \frac{\bar{\theta}}{1-\bar{\theta}} \le \frac{\theta}{1-\theta}$.

We also see from (3.8) that $0 \le \bar{\theta} \le 0.5$. Next notice that (3.6) implies that $\bar{\theta} \ge \theta(1-\bar{\varepsilon})$. Therefore

$$\alpha = \frac{\bar{\theta}}{1-\bar{\theta}} \ge \frac{\theta(1-\bar{\varepsilon})}{1-\bar{\theta}} = \frac{\theta(1-\bar{\varepsilon})}{1-\theta}\frac{1-\theta}{1-\bar{\theta}} = \frac{\theta(1-\bar{\varepsilon})}{1-\theta}\left(1+\frac{\bar{\theta}-\theta}{1-\bar{\theta}}\right) \ge \frac{\theta(1-\bar{\varepsilon})}{1-\theta}\left(1+\frac{\bar{\theta}-\frac{1}{1-\bar{\varepsilon}}\bar{\theta}}{1-\bar{\theta}}\right) =$$

$$= \frac{\theta(1-\bar{\varepsilon})}{1-\theta}\left(1+\frac{\bar{\theta}\frac{-\bar{\varepsilon}}{1-\bar{\varepsilon}}}{1-\bar{\theta}}\right) \ge \frac{\theta(1-\bar{\varepsilon})}{1-\theta}\left(1-\frac{\bar{\varepsilon}}{1-\bar{\varepsilon}}\right) \ge \frac{\theta}{1-\theta}\left(1-\frac{2\bar{\varepsilon}}{1-\bar{\varepsilon}}\right) = \frac{\theta}{1-\theta}\frac{1-3\bar{\varepsilon}}{1-\bar{\varepsilon}}.$$

Proposition 3.4 Let be $\varepsilon \in (0,0.1)$ be given, set $\bar{\varepsilon} = \frac{\varepsilon}{4.1}$, and suppose that Step 1 of the method is executed, with output $(\bar{x},\bar{\theta})$. Then

$$sym(\bar{x},S) \ge sym(S)\frac{1-5\bar{\varepsilon}}{1-\bar{\varepsilon}} \ge (1-\varepsilon)sym(S) \tag{3.9}$$

Proof. Let θ^* denote the optimal objective value of (3.3), and notice that $\bar{\delta} \le \delta^*$ implies that $\theta^* \ge \bar{\theta}^*$. Let $\bar{g}$ be computed in Step 2 of the method. It follows from the stopping criterion in Step 2 that

$$\bar{\theta} \ge (\bar{\theta} + \bar{g})(1-\bar{\varepsilon}) \ge \theta^*(1-\bar{\varepsilon}) \ge \bar{\theta}^*(1-\bar{\varepsilon}) \tag{3.10}$$

From Proposition 3.3 we have

$$sym(\bar{x},S) \ge \frac{\bar{\theta}}{1-\bar{\theta}}\frac{1-3\bar{\varepsilon}}{1-\bar{\varepsilon}} \ge \frac{\bar{\theta}^*}{1-\bar{\theta}^*}\frac{1-\bar{\varepsilon}}{1-\bar{\varepsilon}}\frac{1-3\bar{\varepsilon}}{1-\bar{\varepsilon}} = \frac{\bar{\theta}^*(1-\bar{\varepsilon})}{1-\bar{\theta}^*}\frac{1-3\bar{\varepsilon}}{1-\bar{\varepsilon}}\frac{1-\bar{\theta}^*}{1-\bar{\theta}^*(1-\bar{\varepsilon})} \ge$$

$$\ge sym(S)(1-\bar{\varepsilon})\frac{1-3\bar{\varepsilon}}{1-\bar{\varepsilon}}\frac{1/2}{1-1/2+(1/2)\bar{\varepsilon}} = sym(S)(1-\bar{\varepsilon})\frac{1-3\bar{\varepsilon}}{1-\bar{\varepsilon}}\frac{1}{1+\bar{\varepsilon}} \ge sym(S)\frac{1-5\bar{\varepsilon}}{1-\bar{\varepsilon}} \ge (1-\varepsilon)sym(S)$$

Proposition 3.5 Let $\varepsilon \in (0,0.1)$ be given and set $\bar{\varepsilon} = \frac{\varepsilon}{4.1}$. Suppose that x^a is a $\beta = \frac{1}{8}$-approximate analytic center of S. Then starting with x^a, the stopping criterion of each linear program in Step 1 of method will be reached in no more than $\left\lceil \left(2+4\sqrt{m}\right)\ln \frac{42m}{\varepsilon} \right\rceil$ iterations of Newton's method.

Proof. Step 1 of method is used to approximately solve each of the linear programs (3.1) for $i = 1, \ldots, m$. Let us fix a given i. We will apply to (3.1) a standard feasible path-following interior - point method. A triplet (x, s, z) together with a path parameter μ is a β-approximate solution for μ for the linear program (3.1) if the following system is satisfied

$$\begin{cases} Ax + s = b, s > 0 \\ A^T z = -A_i \\ \left\| \frac{1}{u} Sz - e \right\| \leq \beta \end{cases} \tag{3.11}$$

Now let x^a denote the given $\beta = \frac{1}{8}$-approximate analytic center of the system $Ax \leq b$.

Then there exists (or it is easy to compute) multipliers z^a together with slacks $s^a > 0$ that satisfy the following system:

$$\begin{cases} Ax^a + s^a = b \\ A^T z^a = 0 \\ \left\| S^a z^a - e \right\| \leq \frac{1}{8} \end{cases} \tag{3.12}$$

Define:

$$\left(x^0, s^0, z^0, \mu^0\right) = \left(x^a, s^a, 8s_i^a z - e^i, 8s_i^a\right) \tag{3.13}$$

where e^i is the i^{th} unit vector in R^m. It is then straightforward to show that (3.13) is an $\frac{1}{4}$-approximate solution of (3.11) for the parameter μ^0, so we can start the interior - point method with (3.13). We next bound the value of the parameter μ when the stopping criterion is achieved. Let $(\bar{x}, \bar{s}, \bar{z}, \bar{\mu})$ denote the values of (x, s, z, μ) when the algorithm stops. To keep the analysis simple, we assume that the stopping criterion is met exactly. We therefore have that: $\frac{5}{4} m\bar{\mu} \geq \bar{g} = \bar{\varepsilon}\left(b_i - A_i \bar{x}\right) = \bar{\varepsilon}\bar{s}_i$ which leads to the ratio bound: $\frac{\mu^0}{\mu} \leq \frac{8ms_i^a}{(4/5)\bar{\varepsilon}\bar{s}_i}$.

However, noting that:

$\bar{s}_i = b_i - A_i\bar{x} \geq b_i + \delta_i^* - \bar{g} = b_i + \delta_i^* - \bar{\varepsilon}\bar{s}_i \geq b_i - A_i x^a - \bar{\varepsilon}\bar{s}_i = \bar{s}_i - \bar{\varepsilon}\bar{s}_i$, we obtain $s_i^a \leq (1+\bar{\varepsilon})\bar{s}_i$, and substituting this into the ratio bound yields: $\frac{\mu^0}{\mu} \leq \frac{10m(1+\bar{\varepsilon})}{\bar{\varepsilon}} \leq \frac{42m}{\varepsilon}$

so $\ln\frac{\mu^0}{\mu} \leq \ln\frac{42m}{\varepsilon}$ and so $\left(2+4\sqrt{m}\right)\ln\frac{\mu^0}{\mu} \leq \left(2+4\sqrt{m}\right)\ln\frac{42m}{\varepsilon}$, using $\varepsilon \leq 0.1$, $\bar{\varepsilon} = \frac{\varepsilon}{4.1}$ and Theorem 1.

Proposition 3.6 Let $\varepsilon \in (0,0.1)$ be given, $m \geq 3$ and set $\bar{\varepsilon} = \frac{\varepsilon}{4.1}$. Suppose that x^a is an $\beta = \frac{1}{8}$-approximate analytic center of S. Then starting with x^a, the stopping criterion of each linear program in Step 2 of method will be reached in no more than $\left\lceil \left(2+4\sqrt{m}\right)\ln\frac{6m}{\varepsilon} \right\rceil$ iterations of Newton's method.

Proof. The proof is analogous with proposition 3.6.

Proposition 3.7 Let $\varepsilon \in (0,0.1)$ be given and set $\bar{\varepsilon} = \frac{\varepsilon}{4.1}$. Suppose that x^a is an $\beta = \frac{1}{8}$-approximate analytic center of S. Then starting with x^a and using a standard feasible interior - point method to solve each of the linear programs in Steps 1 and 2, the method will compute an ε-approximate symmetry point of S in no more than $m\left\lceil \left(2+4\sqrt{m}\right)\ln\frac{42m}{\varepsilon} \right\rceil + \left\lceil \left(2+4\sqrt{m}\right)\ln\frac{6m}{\varepsilon} \right\rceil$ total iterations of Newton's method.

Proof. The propositions 3.3 and 3.4 show that the method indeed computes an ε-approximate symmetry point of S. From the propositions 3.5 and 3.6 it follows that the total number of Newton steps computed by the method is no more than $m\left\lceil \left(2+4\sqrt{m}\right)\ln\frac{42m}{\varepsilon} \right\rceil + \left\lceil \left(2+4\sqrt{m}\right)\ln\frac{6m}{\varepsilon} \right\rceil$ since $m \geq n+1 \geq 3$ and $\varepsilon \in (0,0.1)$.

BIBLIOGRAPHY

[1] B. Grűnbaum (1963), "Measures of Symmetry for Convex Sets, in Convexity", Proceedings of Symposia in Pure Mathematics 7, American Mathematical Society, Providence, pp. 233-270;

[2] P. C. Hammer (1951), "The centroid of a convex body", Proc. Amer. Math. Soc. 5 pp. 522-525;

[3] H. Minkowski (1911), "Allegemeine Lehzatze űber konvexe Polyeder", Ges. Abh., Berlin,Vol.2, pp 103-121;

[4] Y. Nesterov, A. Nemirovskii (1994), "Interior-Point Polynomial Algorithms in Convex Programming", SIAM, Philadelphia;

[5] J. Renegar (2001), “A Mathematical View of Interior - point methods in Convex Optimization”, MPS-SIAM Series on Optimization;
[6] C. Roos, T. Terlaky, J.-Ph. Vial (1997), “Theory and Algorithms for Linear Optimization: An Interior Point Approach”, Wiley;
[7] P. Tseng (1992), “Complexity Analysis of a Linear Complimentarily Algorithm Based on a Lyapunov Function”, Mathematical Programming 53 pp. 297-306.